Lacunary Interpolation By Spline Function(0,3)Case

Rostam K. S and Karwan H.J
Department of Mathematics- College of Science- University of Salahaddin-Erbil, Kurdistan Region Iraq
Department of Mathematics- College of Science-University of Sulaimani Kurdistan Region Iraq

Abstract

The object of this paper is to obtain the existence, uniqueness and error bounds of quintic deficient splines interpolating the data (0,3).

Keywords: - *lacunary interpolation.*

1.Introduction

Meir, A. and Sharma, A.[3], Prased, J. and Varma, A. K.[4], Saeed, R. K.[5], Swartz. B. K.and Varma, A. K.[8] have studied the (0,3) lacunary interpolation problem by using spline function. In this paper we consider a related problem, but the essential difference here being in the interpolation point because [1-9] used $t_i = \frac{1}{2}(X_i + X_{i+1})$ but we use $t_{2i} = X_{2i} + \frac{2}{3}h$, i=0, 1, ..., m and 2mh=1. For its description, we shall denote by $S_P(5,3,n)$ the family of five degree splines $S_n(X)$ on the interval [0,1] satisfying the following two properties:

$$
(1.0)\quad \begin{cases} S_n(X) \in C^3[0,1] \\ S_n(X) \text{ is a polynomial of degree five in each subinterval } [X_{2i}, X_{2i+2}], \text{ where } X_{2i} = \frac{i}{m},\ i=0,1,2,\dots,m \text{ and } n=2m+1. \end{cases}
$$

Our method of proof leads to an interesting algorithm for the numerical evalution of the interpolatory splines of the given data. We now state our main results:

Theorem1.1

For given arbitrary numbers $f(X_0)$, $f(X_2)$, ..., $f(X_{2m})$; $f(t_0), f(t_2)$, ..., $f(t_{2m-2})$; $f'''(t_0), f'''(t_2), \dots, f'''(t_{2m-2})$; $f''(X_0), f''(X_{2m})$ there exists a unique spline $S_n(X) \in Sp(5,3,n)$ Such that

$$
(1.1)\quad \begin{cases} \text{(i)}\quad S_n(X_{2i}) = f(X_{2i}),\ i=0,1,\dots,m \\ \text{(ii)}\quad S_n^{(r)}(t_{2i}) = f^{(r)}(t_{2i}),\ i=0,1,\dots,m-1;\ r=0,3 \\ \text{(iii)}\quad S_n''(X_0) = f''(X_0),\quad S_n''(X_{2m}) = f''(X_{2m}) \end{cases}
$$

where

$t_{2i} = X_{2i} + \frac{2}{3}h$, $i = 0,1,\dots,m-1$ and $2mh = 1$.

Theorem 1.2

Let $f \in C^4[0,1]$ and $S_n(X) \in Sp(5,3,n)$ be a unique splines satisfying the conditions of Theorem 1.1. Then

$$\left\| S_n^{(r)}(X) - f^{(r)}(X) \right\| \le 59.5 m^{r-4} w(f^{(4)}; \frac{1}{m}),$$

$r = 0, 1, 2, 3, 4$ and $w(f^{(4)}; \frac{1}{m})$ denotes the modulus of continuity of $f^{(4)}$.

2. Preliminaries

If $Q(X)$ is a polynomial of degree five on $[0,1]$, then we have

(2.0) $$Q(X) = Q(0)\,\lambda_0(X) + Q(\tfrac{1}{3})\lambda_1(X) + Q(1)\lambda_2(X)$$
$$Q''(0)\lambda_3(X) + Q''(1)\lambda_4(X) + Q'''(\tfrac{1}{3})\lambda_5(X)$$

where

(2.1) $$\lambda_0(X) = \frac{1}{10}(27X^5 - 90X^4 + 90X^3 - 37X + 10)$$

(2.2) $$\lambda_1(X) = \frac{1}{20}(-81X^5 + 270X^4 - 270X^3 + 81X)$$

(2.3) $$\lambda_2(X) = \frac{1}{20}(27X^5 - 90X^4 + 90X^3 - 7X)$$

(2.4) $$\lambda_3(X) = \frac{1}{120}(-63X^5 + 180X^4 - 170X^3 + 60X^2 - 7X)$$

(2.5) $$\lambda_4(X) = \frac{1}{120}(9X^5 - 10X^3 + X)$$

(2.6) $$\lambda_5(X) = \frac{1}{120}(-33X^5 + 80X^4 - 50X^3 + 3X)$$

For the later references we note that

(2.7)

$$\lambda_0'(0) = -\frac{37}{10},\ \lambda_0'(1) = \frac{4}{5},$$
$$\lambda_0^{(4)}(0) = -216,\ \lambda_0^{(4)}(\tfrac{1}{3}) = -108,$$
$$\lambda_0^{(4)}(1) = 108,\ \lambda_0'''(0) = 54,\ \lambda_0'''(1) = 0$$
$$\lambda_1'(0) = \frac{81}{20},\ \lambda_1'(1) = -\frac{27}{10},$$
$$\lambda_1^{(4)}(0) = 324,\ \lambda_1^{(4)}(\tfrac{1}{3}) = 162,$$
$$\lambda_1^{(4)}(1) = -162,\ \lambda_1'''(0) = -81,$$
$$\lambda_1'''(1) = 0,\ \lambda_2'(0) = -\frac{7}{20},\ \lambda_2'(1) = \frac{19}{10},$$
$$\lambda_2^{(4)}(0) = -108,\ \lambda_2^{(4)}(\tfrac{1}{3}) = -54,$$
$$\lambda_2^{(4)}(1) = 54,\ \lambda_2'''(0) = 27,\ \lambda_2'''(1) = 0$$
$$\lambda_3'(0) = -\frac{7}{120},\ \lambda_3'(1) = \frac{1}{15},$$
$$\lambda_3^{(4)}(0) = 36,\ \lambda_3^{(4)}(\tfrac{1}{3}) = 15,$$
$$\lambda_3^{(4)}(1) = -27,\ \lambda_3'''(0) = -\frac{17}{2},$$
$$\lambda_3'''(1) = -4,$$
$$\lambda_4'(0) = \frac{1}{120},\ \lambda_4'(1) = \frac{2}{15},$$
$$\lambda_4^{(4)}(0) = 0,\ \lambda_4^{(4)}(\tfrac{1}{3}) = 3,\ \lambda_4^{(4)}(1) = 9$$
$$\lambda_4'''(0) = -\frac{1}{2},\ \lambda_4'''(1) = 4$$
$$\lambda_5'(0) = \frac{1}{40},\ \lambda_5'(1) = \frac{1}{15},$$
$$\lambda_5^{(4)}(0) = 16,$$
$$\lambda_5^{(4)}(\tfrac{1}{3}) = 5,\ \lambda_5^{(4)}(1) = -17$$
$$\lambda_5'''(0) = -\frac{5}{2},\ \lambda_5'''(1) = -3$$

Moreover for $f \in C^4[0,1]$, we have the following expansion

(2.8)

$$f(X_{2i+2}) = f(X_{2i}) + 2hf'(X_{2i}) + \frac{4}{3}h^3 f'''(X_{2i}) + \frac{2}{3}h^4 f^{(4)}(\varepsilon_{1,2i}),$$

$$X_{2i} < \varepsilon_{1,2i} < X_{2i+2}$$

$$f(X_{2i-2}) = f(X_{2i}) - 2hf'(X_{2i}) + 2h^2 f''(X_{2i}) - \frac{4}{3}h^3 f'''(X_{2i}) + \frac{2}{3}h^4 f^{(4)}(\varepsilon_{2,2i}), \quad X_{2i-2} < \varepsilon_{2,2i} < X_{2i}$$

$$f(t_{2i}) = f(X_{2i}) + \frac{2}{3}hf'(X_{2i}) + \frac{2}{9}h^2 f''(X_{2i}) + \frac{4}{81}h^3 f'''(X_{2i}) + \frac{2}{243}h^4 f^{(4)}(\varepsilon_{3,2i}), \quad X_{2i} < \varepsilon_{3,2i} < t_{2i}$$

$$f(t_{2i-2}) = f(X_{2i}) - \frac{4}{3}hf'(X_{2i}) + \frac{8}{9}h^2 f''(X_{2i}) - \frac{32}{81}h^3 f'''(X_{2i}) + \frac{32}{243}h^4 f^{(4)}(\varepsilon_{4,2i}),$$

$$t_{2i-2} < \varepsilon_{4,2i} < X_{2i}$$

$$f''(X_{2i-2}) = f''(X_{2i}) - 2hf'''(X_{2i}) + 2h^2 f^{(4)}(\varepsilon_{5,2i}),$$

$$X_{2i-2} < \varepsilon_{5,2i} < X_{2i}$$

$$f''(X_{2i+2}) = f''(X_{2i}) + 2hf'''(X_{2i}) + 2h^2 f^{(4)}(\varepsilon_{6,2i})$$

$$X_{2i} < \varepsilon_{6,2i} < X_{2i+2}$$

$$f'''(t_{2i-2}) = f'''(X_{2i}) - \frac{4}{3}hf^{(4)}(\varepsilon_{7,2i}),$$

$$t_{2i-2} < \varepsilon_{7,2i} < X_{2i}$$

$$f'''(t_{2i}) = f'''(X_{2i}) + \frac{2}{3}hf^{(4)}(\varepsilon_{8,2i}),$$

$$X_{2i} < \varepsilon_{8,2i} < t_{2i}$$

3. Proof of the Theorem 1.1

The proof depends on the following representation on $S_n(X)$: for $2ih \le X \le (2i+2)h$, $i=0,2,\ldots,m-1$, we have

(3.0)
$$S_n(X) = f(X_{2i})\lambda_0\left(\frac{X-2ih}{2h}\right) + f(t_{2i})\lambda_1\left(\frac{X-2ih}{2h}\right) + f(X_{2i+2})\lambda_2\left(\frac{X-2ih}{2h}\right) + 4h^2 S_n''(X_{2i})\lambda_3\left(\frac{X-2ih}{2h}\right) + 4h^2 S_n''(X_{2i+2})\lambda_4\left(\frac{X-2ih}{2h}\right) + 8h^3 f'''(t_{2i})\lambda_5\left(\frac{X-2ih}{2h}\right).$$

On using (3.0) and

(3.1) $S_n''(0) = f''(0)$, $S_n''(1) = f''(1)$.

It is easy to see that $S_n(X)$ as given by (3.0) indeed satisfies the second condition of (1.0). We still need to decide whether it is possible to determine $S_n''(X_{2i})$, $(i=1,2,\ldots,m-1)$ uniquely. For this purpose we see that the fact that $S_n(X) \in C^3[0,1]$ and therefore the conditions

(3.2) $S_n'''(X_{2i+}) = S_n'''(X_{2i-})$, $i=1, 2, \ldots, m-1$.

With the help of (3.0) and (2.7) reduce to

(3.3)
$$2h^2 S_n''(X_{2i-2}) - \frac{25}{4}h^2 S_n''(X_{2i}) - \frac{1}{4}h^2 S_n''(X_{2i+2}) = -\frac{27}{4}f(X_{2i}) + \frac{81}{8}f(t_{2i}) - \frac{27}{8}f(X_{2i+2}) + \frac{5}{2}h^3 f'''(t_{2i}) - 3h^3 f'''(t_{2i-2})$$

But (3.3) is a strictly tridagonal dominate system then this system can be solved by Gauss elimination method after using Gauss elimination method for solving it , we have unique solution . Thus $S''_n(X_{2i})$, $i=1, 2, ..., m-1$ can be obtained . This proves the Theorem 1.1.

To prove Theorem 1.2 we need the following lemma .

Lemma 4.1

Let $B_{2i} = |f''(X_{2i}) - S''_n(X_{2i})|$, then

$$\max_{1\le i\le m-1} B_{2i} \le \frac{25}{18} h^2 w(f^{(4)};\frac{1}{m}).$$

Proof:

From (3.3) and (2.8) it follow that

$$2h^2(S''_n(X_{2i-2}) - f''(X_{2i-2})) - \frac{25}{4}h^2(S''_n(X_{2i}) - f''(X_{2i}))$$
$$-\frac{1}{2}h^2(S''_n(X_{2i+2}) - f''(X_{2i+2})) =$$
$$-4h^4 f^{(4)}(\varepsilon_{5,2i}) + \frac{1}{2}h^4 f^{(4)}(\varepsilon_{6,2i}) + \frac{1}{12}h^4 f^{(4)}(\varepsilon_{3,2i})$$
$$-\frac{9}{4}h^4 f^{(4)}(\varepsilon_{1,2i}) + \frac{5}{3}h^4 f^{(4)}(\varepsilon_{8,2i}) +$$
$$4h^4 f^{(4)}(\varepsilon_{7,2i}) = \frac{25}{4}\theta h^4 w(f^{(4)};\frac{1}{m}), |\theta| \le$$

By using the property of diagonal dominates we have

$$\max_{1\le i\le m-1} B_{2i} \le \frac{25}{18} h^2 w(f^{(4)};\frac{1}{m}).$$

Lemma 4.2

For $f \in C^4[0,1]$,then

(i) $|S_n^{(4)}(X_{2i+}) - f^{(4)}(X_{2i})| \le 18 w(f^{(4)};\frac{1}{m})$

(ii) $|S_n^{(4)}(X_{2i-}) - f^{(4)}(X_{2i})| \le \frac{85}{3} w(f^{(4)};\frac{1}{m})$

(iii) $|S_n^{(4)}(t_{2i}) - f^{(4)}(t_{2i})| \le \frac{19}{2} w(f^{(4)};\frac{1}{m})$

Proof of (i)

On using (3.0), (2.7) and (2.8) we obtain

$$h^4 S_n^{(4)}(X_{2i+}) = -\frac{27}{2} f(X_{2i}) + \frac{81}{4} f(t_{2i}) - \frac{27}{4} f(X_{2i+2}) +$$
$$8h^2 S''_n(X_{2i}) + 8h^3 f'''(t_{2i})$$

Hence

$$h^4(S_n^{(4)}(X_{2i+}) - f^{(4)}(X_{2i})) =$$
$$\frac{1}{6}h^4 f^{(4)}(\varepsilon_{3,2i}) - \frac{9}{2}h^4 f^{(4)}(\varepsilon_{1,2i})$$
$$+9h^2(S''_n(X_{2i}) - f''(X_{2i})) +$$
$$\frac{16}{3}h^4 f^{(4)}(\varepsilon_{8,2i}) - f^{(4)}(X_{2i})$$
$$= \frac{11}{2}h^4\theta_1 w(f^{(4)};\frac{1}{m}) + 9h^2(S''_m(X_{2i}) - f''(X_{2i})), |\theta_1| \le$$

On using Lemma 4.1, the result (i) follows .The proof of (ii) and (iii) are similar .We only mention that

$$h^4 S_n^{(4)}(X_{2i-}) = \frac{27}{4} f(X_{2i-2}) - \frac{81}{8} f(t_{2i-2}) +$$
$$\frac{27}{8} f(X_{2i}) - \frac{27}{4} h^2 S''_n(X_{2i-2}) +$$
$$\frac{9}{4} h^2 S''_n(X_{2i}) - \frac{17}{2} h^3 f'''(t_{2i-2}),$$

and

$$h^4 S_n^{(4)}(t_{2i}) = -\frac{27}{4} f(X_{2i}) + \frac{81}{8} f(t_{2i}) - \frac{27}{8} f(X_{2i+2}) + \frac{15}{4} h^2 S_n''(X_{2i}) + \frac{3}{4} h^2 S_n''(X_{2i+2}) + \frac{5}{2} h^3 f'''(t_{2i}).$$

5. Proof of the Theorem 1.2

Since $0 \le X \le 1$ we have

(5.0) $\lambda_0(X) + \lambda_1(X) + \lambda_2(X) = 1$

Let $X_{2i} \le X \le X_{2i+2}$ On using (5.0) and (3.0) we obtain

$$S_n^{(4)}(X) - f^{(4)}(X) = (S_n^{(4)}(X_{2i+}) - f^{(4)}(X))\lambda_0(\frac{X-2ih}{2h}) \quad (5.1)$$

$$+ (S_n^{(4)}(X_{2i+2}) - f^{(4)}(X))\lambda_2(\frac{X-2ih}{2h}) + (S_n^{(4)}(t_{2i}) - f^{(4)}(X))\lambda_1(\frac{X-2ih}{2h})$$

$$= L_1 + L_2 + L_3$$

Where

(5.2) $L_1 = (S_n^{(4)}(X_{2i+}) - f^{(4)}(X))\lambda_0(\frac{X-2ih}{2h})$

(5.3) $L_2 = (S_n^{(4)}(X_{2i+2}) - f^{(4)}(X))\lambda_2(\frac{X-2ih}{2h})$

(5.4) $L_3 = (S_n^{(4)}(t_{2i}) - f^{(4)}(X))\lambda_1(\frac{X-2ih}{2h})$

From (2.1) –(2.3), one can easily see that

$$(5.5) \quad \begin{cases} |\lambda_0(X)| \le 1 \\ |\lambda_1(X)| \le 1 \\ |\lambda_2(X)| \le 1 \end{cases}$$

for $0 \le X \le 1$

On using lemma4.2 and (5.5) in (5.2) – (5.4) we obtain

$$|L_1| \le 19 w(f^{(4)}; \frac{1}{m}), \quad |L_2| \le \frac{88}{3} w(f^{(4)}; \frac{1}{m}),$$

and $|L_3| \le \frac{21}{2} w(f^{(4)}; \frac{1}{m})$.

Using above result in (5.1) we obtain

$$|S_n^{(4)}(X) - f^{(4)}(X)| \le 59 w(f^{(4)}; \frac{1}{m})$$

This prove Theorem1.2 for r=4. Since

$$S_n^{(3)}(X) - f^{(3)}(X) = \int_{t_{2i}}^{x} (S_n^{(4)}(t) - f^{(4)}(t))\,dt$$

and because $S_n^{(3)}(t_{2i}) - f^{(3)}(t_{2i}) = 0$ by (1.1) then we obtain

$$|S_n^{(3)}(X) - f^{(3)}(X)| \le 59 m^{-1} w(f^{(4)}; \frac{1}{m})$$

which proves Theorem1.2 for r=3. To prove Theorem1.2 for r=2 . Since

$$S_n^{(2)}(X) - f^{(2)}(X) = \int_{t_{2i}}^{x} (S_n^{(3)}(t) - f^{(3)}(t))\,dt + S_n''(t_{2i}) - f''(t_{2i}).$$

On using above result and the fact

$$|S_n^{(2)}(t_{2i}) - f^{(2)}(t_{2i})| \le \frac{61}{54} h^2 w(f^{(4)}; \frac{1}{m})$$

we obtain

$$|S_n^{(2)}(x) - f^{(2)}(X)| \le 59.5 m^{-2} w(f^{(4)}; \frac{1}{m})$$

Which proves Theorem1.2 for $r=2$, the proof of Theorem1.2 for $r=0, 1$ follows immediately on the line of Saeed , R.K. [5]. This completes the proof of Theorem 1.2.

Conclusion

In this paper we conclude that the end conditions have to play a very important role in deciding the rate of convergence, but the type of lacunary data has no any role in deciding the rate of convergence.

References

[1]Bokhari, M. A., Dikshit, H. P. and Sharma, A: Birkhoff interpolation on some perturbed roots of unity, *Numerical Algorithms* 2000, **25**(1/4) 47-62

[2]Lenard, M.: Modified (0,2)-interpolation on the roots of Jacobi polynomials, I(Explicit formulae), *Acta Math. Hungar* 1999, **84**(4) 317-327.

[3]Meir, A. and Sharm, A. : Lacunary interpolation by splines , *SIAM J. Nume. Anal.* 1973, **10** (3) 433-442.

[4]Prasad, J. and Varma, A. K. : Lacunary interpolation by quintic splines, *SIAM J. Nume. Anal.1979*,**16** (6) 1075-1079.

[5]Saeed, R. K. : Study of Launacry Interpolation by splines, *M. Sc. thesis,* Unversity of Salahaddin ,College of Scince,(1990).

[6]Sharma,A., Szabados, J., Underhill, B. and Varma, A. K.: On some general lacunary interpolation problems, *J. Approx. Theory* 1996, 87(2), 194-219.

[7]Swartz, B. K: and Varga, R. S. : A note on Lacunary Interpolation by Splines, *SIAM J. Nume. Anal.* 1973,**10** (1) 443-447.

[8]Varma,A.K. : Lacunary Interpolation by Splines,I,*Acta. Math.Acad.Sci.Hungar* 1987 ,**31** (34) 185-192.

[9]Varma,A.K. :Lacunary Interpolation by Splines,II,*Acta. Math.Acad.Sci. Hungar* 1987, **31** (34) 193-203.

لەناو یەکدانانی بۆشایی بە یارمەتی نەخشەی سپلاینی پلە پێنج- باری (0,3)

روستم کریم سعید وکاروان حمەفرج جوامیر

بەشی ماتماتیك / کۆلیجی زانست / زانکۆی سەلاحەددین /هەولێر – هەرێمی کوردستان- عێراق

بەشی ماتماتیك / کۆلیجی زانست/ زانکۆی سلیمانی – هەرێمی کوردستان- عێراق

پوختە

لەم توێژینەوەیەدا بوون و یەکایەتی و راددەی هەڵەمان بەدەست هێنا بۆ نەخشەی سپلاینی پلە پێنج کە شیکارە بۆ داتاکانی بۆشایی (0,3).

الاندراج الفراغي بواسطة دالة سبلاين-حالة (0,3)

روستم كريم سعيد و كاروان حمه فرج جوامير

قسم الرياضيات/ كلية العلوم / جامعة صلاح الدين- أربيل/ اقليم كردستان - العراق

قسم الرياضيات/ كلية العلوم/ جامعة السليمانية/ اقليم كردستان - العراق

الخلاصة

حصلنا في بحثنا هذا على وجود و وحدانية و حدود الخطأ لدالة سبلاين من الدرجة الخامسة والتى هي حل للبيانات الفراغية (0,3).

Received 6 /3/2002 .Accepted 16/9/2002.   وەرگیرا لە ٢٠٠٢/٣/٦دا وەپەسەندکرا لە ٢٠٠٢/٩/١٦.